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Theory of the voltage biased Josephson model

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Abstract. The voltage biased weak link between two bulk superconductors is discussed using the conventional Josephson pendulum model. A parameter λ is introduced to characterize the ratio between the electron pair tunnelling energy and the electrostatic energy required to store one electron pair in the link. If λ is small, the Q–V curve, the relationship between the applied voltage and the charge stored in the link, is in a staircase form. When there are strong pair tunnelling processes, $\lambda \gg 1$, the Q–V curve becomes a straight line. These results are consistent with experimental findings.

It was found experimentally (Prance *et al* 1985) that in a voltage biased weak link between two bulk superconductors, the Q–V behaviour, i.e. the relationship between the applied voltage V and the stored charge Q, develops in a staircase form. As the tunnelling energy of the sample increases, the staircase form changes into a smooth line. This interesting phenomenon has not had any theoretical explanation as yet. Since in a voltage biased weak link there is an AC current but no DC current, a naive conjecture may easily follow: the average charge stored in the link can be hardly affected by the tunnelling energy. To a certain extent, the above finding remains a puzzle.

In the present paper, we shall use Josephson pendulum model to investigate this issue. The Josephson pendulum model has long served as a canonical model for the discussion of weak links (Josephson 1962, Barone and Paterno 1982, Anderson 1967, Feynman and Vernon 1963, Solymar 1972). We have found that there is a dimensionless parameter λ,

$$\lambda = \sqrt{\hbar\nu/(q^2/C)} = \sqrt{\hbar\nu C/q^2} \tag{1}$$

where $\hbar\nu$ is the electron pair tunnelling energy, C the capacitance of the link, and $q = 2e$. It is clear that λ characterizes the ratio between the electron pair tunnelling energy and the electrostatic energy required to store one electron pair in the weak link. When λ is small, the Q–V behaviour develops into a staircase form. When there are strong pair tunnelling processes, $\lambda \gg 1$, the Q–V curve becomes a straight line, similar to a classical capacitor. Our theoretical result is consistent with the reported experiment (Prance *et al* 1985) where the staircase form is found at $\lambda = 0.22$.

Recall the Hamiltonian of the quantum mechanical Josephson pendulum, $H = Q^2/2C - \hbar\nu\cos\theta$, where θ is the phase difference between two bulk superconductors. The charge stored capacitively is $Q = -qN$ where N is the number of electron pairs stored

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in the link. We assume that the two bulk superconductors are identical and have N_0 electron pairs when no voltage bias is applied. After a voltage V is applied, the bulk superconductor serving as a positive plate in the capacitor has phase θ_1 and N_1 electron pairs, while the other has phase θ_2 and N_2 electron pairs. The phase difference of the link is $\theta = \theta_2 - \theta_1$. Since the phases of bulk superconductors, θ_1 and θ_2, are periodic in 2π, θ is also periodic in 2π. Following Anderson (Anderson and Dayem 1964), we can write N_j as an operator $N_j = -\mathrm{i}\,\partial/\partial\theta_j\,(j = 1, 2)$. As in the case of a capacitor, if one plate has charge Q, the other plate has charge $-Q$. Therefore, $Q = -q(N_1 - N_0)$ and $-Q = -q(N_2 - N_0)$. Hence, $N = \frac{1}{2}(N_2 - N_1)$. Define $\theta_0 = (\theta_1 + \theta_2)/2$. Using θ_0 and θ to replace θ_1 and θ_2, we have $N_1 + N_2 = -\mathrm{i}\,\partial/\partial\theta_1 - \mathrm{i}\,\partial/\partial\theta_2 = -\mathrm{i}\,\partial/\partial\theta_0$ and

$$N = -(\mathrm{i}/2)(\partial/\partial\theta_2 - \partial/\partial\theta_1) = -\mathrm{i}\,\partial/\partial\theta. \tag{2}$$

Then we have

$$H = -(q^2/2C)(\partial/\partial\theta)^2 - \hbar\nu\cos\theta. \tag{3}$$

The voltage bias across the weak link may (in the model under discussion) be viewed as an electron pair chemical potential difference between the bulk superconductors,

$$\mu = qV \tag{4}$$

so that the grand canoical Hamiltonian can be employed, i.e.

$$H(V) = H - \mu N. \tag{5}$$

Equations (2), (3), (4), and (5) imply the voltage biased pendulum Hamiltonian given by

$$H(V) = -(q^2/2C)(\partial/\partial\theta)^2 + \mathrm{i}qV(\partial/\partial\theta) - \hbar\nu\cos\theta. \tag{6}$$

As stated earlier, θ is periodic in 2π. The wave function of the above Hamiltonian is also periodic

$$\psi(\theta + 2\pi) = \psi(\theta). \tag{7}$$

Conventionally, the Josephson pendulum model is treated classically. The theoretical properties of the model in a fully quantum mechanical form have not been explored yet. Therefore, before we proceed any further, we first want to demonstrate that the quantum mechanical treatment can produce all the results the classical treatment provides.

We first consider a DC pair current in the weak link. The electron pair current operator is given by

$$I = q[N, H]/(\mathrm{i}\hbar) = -q\nu\sin(\theta). \tag{8}$$

Let Ψ_0 be the ground state of the grand canonical Hamiltonian. The DC current is given by

$$\langle\Psi_0|I|\Psi_0\rangle = (\mathrm{i}q/)\hbar)\langle\Psi_0, [H(V), N]\Psi_0\rangle = 0. \tag{9}$$

This implies that the quantum treatment leads to a vanishing average DC current in a voltage biased weak link, a conclusion consistent with the result derived by Büttiker (1987).

To discuss the AC pair current, we first examine the parameter λ. For most weak links, λ is very big. For example, most Josephson junctions have $\hbar\nu$ of the order of 0.1 eV. Then, we have $\lambda^2 \sim 10^6$ for the capacitance $C \sim 1pf$. If C is larger, λ is even bigger. In the case of $\lambda \gg 1$, we can approximate the Hamiltonian in (6) by ignoring the electrostatic energy. Then the Schrödinger equation ($C \to \infty$) reduces to

$$(\mathrm{i}qV\partial/\partial\theta - \hbar\nu\cos\theta)\Psi = E\Psi. \tag{10}$$

The eigenstates have the form

$$\Psi(\theta) = \exp(-\mathrm{i}E\theta/qV - \mathrm{i}\hbar\nu\sin\theta/qV). \tag{11}$$

Since it is required that $\Psi(\theta + 2\pi) = \Psi(\theta)$, from (11) we have $E = qVn$ where n is an integer. The time-dependent wave function is given by

$$\Psi(\theta, t) = \sum_n c_n \exp\left(-\mathrm{i}n\theta - \frac{\mathrm{i}\hbar\nu\sin\theta}{qV} - \frac{\mathrm{i}nqVt}{\hbar}\right). \tag{12}$$

Then the AC pair current

$$I = \langle\Psi| - q\nu\sin\theta|\Psi\rangle \tag{13}$$

is strictly time periodic with frequency qV/h, a conclusion consistent with the well-known classical treatment.

The progress of technology in relation to the manufacture of weak links has made possible the provision of samples of small λ. For example, in the work of Prance *et al* (1985), the weak link has $\hbar\nu \approx 0.05q^2/C$ which yields $\lambda \approx 0.22$. Therefore, the case of finite λ is practical and interesting to study. We note that the classical treatment is absent in such a case. Especially, from a point of view of classical physics, the charge stored in the link, qN, can be hardly affected by the tunnelling energy, $\hbar\nu$, because the tunnelling does not produce any DC current. However, the experiment has found that the Q–V behaviour changes as the tunnelling energy $\hbar\nu$ varies. We note that from (1) that a small λ can be realized at small C. Recently, low-capacitance Josephson junctions (Büttiker 1987, Widom *et al* 1982) and small normal tunnel junctions (Ben-Jacob and Gefen 1985) have been found to have quantum mechanical effect and have received renewed interest. We, therefore, believe that the quantum mechanical treatment of the present problem is also needed.

Consider the Hamiltonian in (6). When λ is finite, the capacitance energy is compatible with the tunnelling energy, whose role cannot be ignored. For $H(V)$ in (6) with a finite λ, there is no simple analytical form available for the eigenstates. We write the ground state in the form

$$\Psi = \sum_{n=-\infty}^{\infty} c_n \,\mathrm{e}^{\mathrm{i}n\theta}. \tag{14}$$

Then, the Schrödinger equation $H(V)\Psi = E\Psi$ reduces to a tridiagonal matrix eigenvalue equation

$$-\lambda^2 c_{n-1} + c_n(n^2 - 2nCV/q - 2CE/q^2) - \lambda^2 c_{n+1} = 0. \tag{15}$$

We numerically diagonalize the above equation to find the ground state. After having found the ground state, we calculate the charge stored in the weak link,

$$Q = \langle\Psi_0|qN|\Psi_0\rangle = q \sum_{n=-\infty}^{\infty} n|c_n|^2 \Big/ \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{16}$$

From (15), CE/q^2 and c_n are functions of CV/q and λ. Then the Q–V relationship can be expressed in an equation of state

$$Q = qf(CV/q, \lambda) \tag{17}$$

where the $f(x, \lambda)$ is a numerically computed function. Our numerical results are plotted

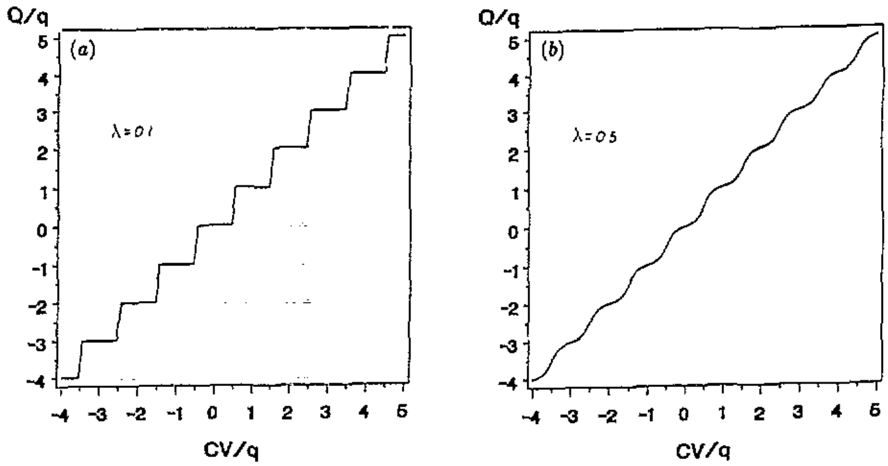

Figure 1. (a) Q as a function of V for $\lambda = 0.1$; staircase behaviour is clear. (b) The charge storage plotted for $\lambda = 0.5$; the curve is much smoother.

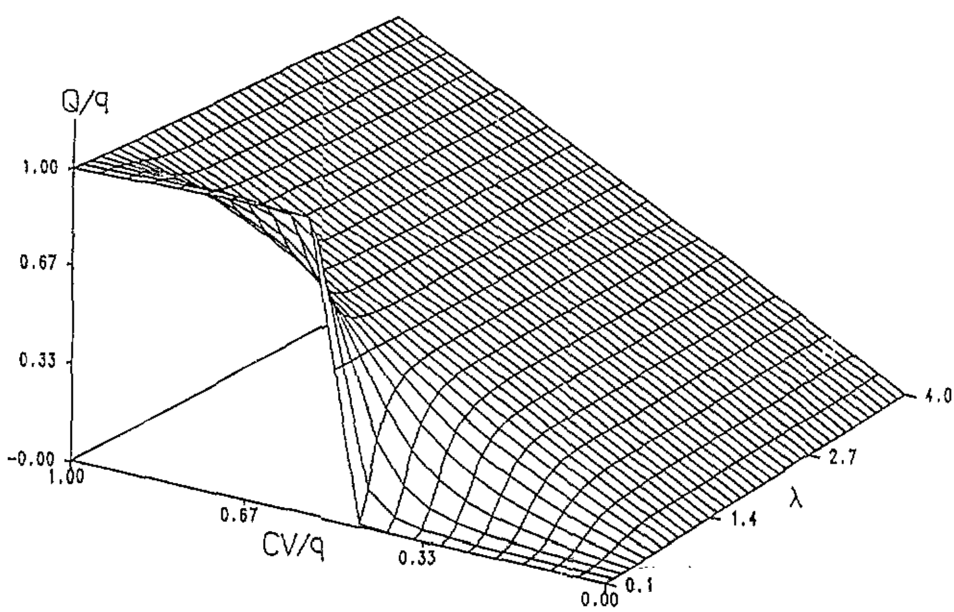

Figure 2. The function $Q/q = f(CV/q, \lambda)$ is plotted in the three dimensional form. As λ increases, the Q–V curve changes from a staircase form to a straight line.

in figures 1 and 2. As shown in figure 1(a), for a small λ ($\lambda = 0.1$), the electron pair tunnelling processes are relatively weak and the Q–V relationship, i.e. the equation of state (17), develops into periodic steps. This result is not difficult to understand. For very small λ, we can use perturbation to derive the ground state from (15). The leading term is $c_k\,e^{ij\theta}$ with $k = [CV/q]$ where the notation of $[y]$ refers to the integer closest to y. The next two correction terms are $c_{k+1}\,e^{i(k+1)\theta}$ and $c_{k-1}\,e^{i(k-1)\theta}$. Both c_{k+1} and c_{k-1} have the order λ^2. Then as CV/q varies, the leading term of the charge $Q/q \approx k$, showing a staircase behaviour. The correction term to Q/q is of the order of λ^2.

With an increase of the tunnelling energy, the quantum mechanical effect becomes more and more important. As a comparison, in figure 1(b) λ increases to 0.5, the Q–V curve is much smoother than in figure 1(a). This behaviour is difficult to explain using the classical treatment. Quantum mechanically, this change reflects the fact that as λ increases, more and more states $e^{in\theta}$ in (14) with different number of pairs, n, have a compatible contribution to the ground state. For $\lambda \gg 1$, there are strong pair tunnelling processes. However, Q is now is averaged over a large number of states $e^{in\theta}$, Q becomes smooth and the weak link behaves as a classical capacitor with the equation of state $Q = CV$. We plot the detail of the change in Q–V behaviour against λ in a three-dimensional graph (figure 2).

Our results in figures 1 and 2 are qualitatively consistent with the experimental result (Prance *et al* 1985). The change of the Q–V curve as a function of λ is also observed in the experiment. For a quantitative comparison, some parameters in the experiment, such as capacitance C, should be accurately determined and more experiments are needed.

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References

Anderson P W and Dayem A H 1964 *Phys. Rev. Lett.* **13** 195
Anderson P W 1967 *Progress in Low Temperature Physics* vol V, ed C J Gorter (Amsterdam: North-Holland) pp 1–43
Barone A and Paterno G 1982 *Physics and Applications of the Josephson Effect* (New York: Wiley)
Ben-Jacob E and Gefen Y 1985 *Phys. Lett.* **108A** 289
Büttiker M 1987 *Phys. Rev.* B **36** 3528
Feynman R P and Vernon F L 1963 *Ann. Phys., NY* **24** 118
Josephson B D 1962 *Phys. Lett.* **1** 251
Prance H, Prance R J, Spiller S P, Mutton J E, Clark T C 1985 *Phys. Lett.* **111A** 199
Solymar L 1972 *Superconductive tunnelling and applications* (London: Chapman and Hall) ch 14
Widom A, Megaloudis G, Clark T D, Prance H and Prance R J 1982 *J. Phys.* A **15** 3895